Math 53, Discussions 116 and 118 **2022-11-04**

Line integrals

Answers

Questions

Question 1. Determine if each of the following vector fields is conservative. If they are, find a potential function.

(a) $\mathbf{F} = \langle 3x^2 + y^2, -2xy \rangle$
(b) $\mathbf{F} = \langle 3x^2 - y^2, -2xy \rangle$
(c) $\mathbf{F} = \langle 3x^2 + y^2, 2xy \rangle$
(d) $\mathbf{F} = \langle 3x^2 - y^2, 2xy \rangle$

Question 2. Evaluate the line integral $\int_C (\sin x \, \mathrm{d}x + \cos y \, \mathrm{d}y)$, where C consists of the top part of the circle $x^2 + y^2 = 1$ from $(1, 0)$ to $(-1, 0)$, followed by the line segment from $(-1, 0)$ to $(2, -\pi)$.

Question 3. Let C be the portion of the curve $x = y^2/2$ in the range $-2 \le y \le 1$. Evaluate

$$\int_C (y + 2xy) \, \mathrm{d}s.$$

Below are brief answers to the worksheet exercises. If you would like a more detailed solution, feel free to ask me in person. (Do let me know if you catch any mistakes!)

Answers to questions

Question 1. All of these vector fields are defined on $\mathbb{R}^2$. Writing $\mathbf{F} = \langle P, Q \rangle$, we can check whether $\mathbf{F}$ is conservative by seeing whether $Q_x - P_y = 0$.

(a) Not conservative
(b) Conservative. A potential function is $f(x, y) = x^3 - xy^2$.
(c) Conservative. A potential function is $f(x, y) = x^3 + xy^2$.
(d) Not conservative

Question 2. The vector field $\langle \sin x, \cos y \rangle$ is conservative. A potential function is

$$f(x, y) = -\cos x + \sin y.$$

Hence the value of the integral, by FTLI, is

$$(-\cos 2 + \sin(-\pi)) - (-\cos(-1) + \sin 0) = \cos(-1) - \cos 2.$$

Question 3. For integrals with respect to ds, you don't really have any choice other than to compute directly via parametrization (tools such as FTLI are not available). We can parametrize C simply as $x = t^2/2$, $y = t$, $-2 \leq t \leq 1$. The integral becomes

$$\int_{-2}^{1} (t + t^3)\sqrt{t^2 + 1}\, \mathrm{d}t$$

which can be handled by a substitution $u = t^2 + 1$, $\mathrm{d}u = 2t\, \mathrm{d}t$. The final answer is $\frac{4\sqrt{2}}{5} - 5\sqrt{5}$.